1. INTRODUCTION

Exchange-traded funds (ETFs), also known as trackers in Europe, are similar to mutual funds but their shares can be traded at any time on an exchange. The first ETFs have been designed to track an index and, as such, have been passive funds. Since the mid 2000’s, actively managed ETFs have appeared on the market, with leveraged and inverse ETFs designed to achieve multiple exposure (positive or negative, e.g., exposures equal to 2x or -2x the benchmark ETF) to index returns, on a daily basis. The first leveraged and inverse ETFs were launched between June 2006 and June 2009 in the U.S.\(^1\) and are

\(\text{Analysis and Comparison of Leveraged ETFs and CPPI-type Leveraged Strategies}\)

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1. In 2009, the total amount of leveraged ETFs traded in the US financial market was higher than US$31 billions.
become rather popular. More than 150 leveraged and inverse ETFs are currently traded in the U.S., covering a broad range of equity, sector, international, fixed-income, commodity and currency markets.

As mentioned by Charupat and Miu (2011), while the amount of leveraged funds is relatively small compared to the whole ETF market, their trading volume and value are very large (for example, in Canada, during September 2009, the amount of leveraged ETFs was about 9% of the whole ETF market while the percentages of their trading volume and value were respectively equal to 67% and 48%). Ramaswamy (2011) mention also that, “while the assets under management of leverage or inverse ETFs amount to only around $40 billion globally, which is about 3% of ETF assets, they account for nearly 20% of the turnover in ETF assets, suggesting that they are very actively traded.” This is mainly due to the fact that such speculative financial products concern short-term traders with portfolio rebalancing on a daily basis. In France, a leveraged ETF which differed slightly from the US model, was introduced on the market by SGAM AI in October 2005.

Recently, there has been some controversy surrounding leveraged ETFs in the U.S. market, focused mainly on the performance results delivered by these products over extended periods of time. The most frequent criticism has been that leveraged ETFs do not perform as they should or as they are claimed to. The Financial Industry Regulatory Authority (FINRA) published in June 2009 a regulatory notice containing the following : “While such products may be useful in some sophisticated trading strategies, they are highly complex financial instruments that are typically designed to achieve their stated objectives on a daily basis. Due to the effects of compounding, their performance over longer periods of time can differ significantly from their stated daily objective. Therefore, inverse and leveraged ETFs that are reset daily typically are unsuitable for retail investors who plan to hold them for longer than one trading session, particularly in volatile markets.” For the moment, this controversy has not crossed the Atlantic. In Europe, NYSE Euronext publishes leveraged indices (which belong to the family of strategy indices) on several national European equity

2. SGAM stands for “Société Générale Asset Management”. AI means “Alternative Investment”.
3. It also publishes short, double short, triple short and triple leverage indices.
indices (CAC 40, AEX, BEL20, PSI20) which are designed to be the underlying indices for ETFs. It gives the following definition of the double leverage index: “The leverage index tracks the performance of a strategy which doubles exposure to an underlying index with the support of a short-term financing” on a daily basis. For instance, on the French market, on February 10, 2010, ComStage (a subsidiary of Commerzbank) issued a leveraged ETF on the CAC 40 that is aimed at replicating the ETF CAC 40 Leverage index, an index launched on December 21, 2007 by Euronext.

It is worth noting that in France, unlike the U.S., some of the first of these leveraged funds have been managed with a slightly modified Constant Proportion Portfolio Insurance (CPPI) type strategy. Examples include SGAM ETF Leveraged CAC 40 and SGAM ETF Bear CAC 40 launched on October 19, 2005 by SGAM Alternative Investment. The first of these funds is not strictly speaking a leveraged ETF, since the leverage is at most 200% but can be less, depending on market circumstances and also on the expectations of the portfolio manager. In what follows, we analyze these funds in more details. Avellaneda and Zhang (2010), Jarrow (2010) and Giese (2010), have recently studied these financial products mainly on a theoretical basis. Their studies only explore the US case and none of them draws parallels with CPPI-type strategies, nor considers products like that issued by SGAM AI.

This paper is organized as follows. Section 2 presents an introductory example which sets the background of the paper and illustrates differences between standard LETF and Leveraged CPPI strategy in a discrete-time setting. In Section 3, we analyze the value process of a leveraged ETF that corresponds to a convex constant allocation portfolio strategy. We study main of its statistical properties and examine its return as a function of the risky asset return. As a by-product, we show that the stock index price can increase while, at the same time, the leveraged fund decreases. We provide the probability of such puzzling event and detail its main sensitivities to financial and management

4. To the best of our knowledge, leveraged funds have not been analyzed as CPPI portfolio in the US.
5. After a reorganization of the asset management activity of Société Générale, these funds are now managed by LYXXOR since the 1st of September 2009.
6. See the “prospectus complet” of the SGAM ETF Leveraged CAC 40, certified by the “Autorité des marchés financiers” (AMF).
parameters. In the continuous-time framework, we prove an equivalence result stating that a leveraged ETF can be also be viewed as a CPPI fund with a floor proportional to the portfolio value itself. In Section 4, we compare Leveraged ETFs and specific Leveraged CPPI strategies\(^7\) in a more practical framework. First, we determine a quasi-explicit expression of the Leveraged CPPI value, taking account of the discrete-time variations of the leverage level. Then, we conduct comparisons by means of Monte Carlo simulations. Finally, we use Omega and Kappa performance measures to compare both strategies by means of Monte Carlo experiments.

2. INTRODUCTORY EXAMPLE AND MOTIVATIONS

There is a widely held misconception of what exactly leveraged ETFs provide: returns of 2x or 3x the annual return of the underlying market index (which corresponds to a static leveraged ETF)? To date, leveraged ETFs have been designed to deliver daily returns that are a positive or negative multiple of the daily return of the underlying index (which corresponds in discrete-time to a daily leveraged ETF). In what follows, we recall briefly the mechanisms of static and daily leveraged ETFs jointly with the standard and leveraged CPPIs.

The static leveraged portfolio strategy is a buy-and-hold strategy that aims to reproduce the leveraged performance of the stock index over a given period without continuous-time rebalancing (for practical purposes, daily). This can be considered as a benchmark for the sake of comparison.\(^8\)

Suppose that the investor trades in discrete-time. Denote by \(X_k\) the arithmetical return of the risky asset between times \(t_{k-1}\) and \(t_k\). We have:

\[
X_k = \frac{S_{t_k} - S_{t_{k-1}}}{S_{t_{k-1}}}. \tag{1}
\]

Let us denote \(V_{0}^{SL}\) the value process of the static leveraged ETF at time 0 where \(L\) is the leverage coefficient of the initial portfolio value.

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7. This latter one is illustrated by using the SGAM Leveraged CAC 40 strategy.
8. See Appendix A where the static leverage case is discussed for the geometric Brownian motion case.
This initial portfolio value is split into two amounts:

\[ V_0^{SL} = LV_0^{SL} + (1 - L)V_0^{SL}, \]

where \( LV_0^{SL} \) is the amount (or “exposure”) invested on the risky asset \( S \) and \((1 - L)V_0^{SL}\) is the amount invested on the riskless asset \( B \). We denote by \( r \) the riskless rate on the time period \([t_k,t_{k+1}]\). Therefore, at any time \( t_k \), the static leveraged portfolio return is given by:

\[ \frac{V_{t_k}^{SL}}{V_0^{SL}} = L \left( \frac{S_k}{S_0} \right) - (L - 1)(1 + r)^k, \]

which implies (for short horizon):

\[ \frac{V_{t_k}^{SL} - V_0^{SL}}{V_0^{SL}} \simeq L \left( \frac{S_k - S_0}{S_0} \right). \]

In that case, the (arithmetic) rate of return of portfolio value is approximately equal to the rate of risky asset return multiplied by the leverage coefficient \( L \).

The daily leveraged ETF is a dynamic strategy the performance of which is calculated every day according to the following equation:

\[ \frac{V_{t_k+1}^{SL}}{V_{t_k}^{SL}} = L \left( \frac{S_{k+1}}{S_k} \right) - (L - 1)(1 + r), \]

The Leveraged CPPI is a simplified strategy to allocate assets dynamically over time. The Leveraged CPPI is based on a constant proportion of the portfolio value invested on the risky asset. For the standard CPPI strategy, this constant proportion is defined on the excess value of the portfolio with a given floor. We describe hereafter these two methods:

- The investor starts by setting a floor \( F_k \) equal to the lowest acceptable value of the portfolio at time \( t_k \). It means that the guarantee

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constraint is to keep the portfolio value $V_k$ above the floor $F_k$ whose dynamics is given by:

$$F_{k+1} = F_k (1 + r) .$$

Then, the investor computes the cushion $C_k$ as the excess of the portfolio value over the floor ($C_k = V_k - F_k$) and determines the amount invested in the risky asset by multiplying the cushion by a predetermined multiple, denoted $m$. The total amount invested in the risky asset is known as the exposure $e_k$ ($e_k = mC_k$). The remaining funds ($V_k - e_k$) are invested in the riskless asset.

Therefore, the dynamic evolution of the portfolio value is given by:

$$V_{k+1} = V_k + e_k X_{k+1} + (V_k - e_k) r ,$$

where the cushion value is defined by:

$$C_{k+1} = C_k \left[ 1 + m X_{k+1} + (1 - m) r \right] ,$$

which implies:

$$C_k = C_0 \prod_{1 \leq l \leq k} \left[ 1 + m X_l + (1 - m) r \right] .$$

A modified version of the standard CPPI method leads to the Leveraged CPPI (LCPPI). This method has a floor that is constant over time: $F_k = F_0 = \xi V_0$, $\forall k$. The parameter $\xi$ corresponds to the initial proportion of the portfolio value that is designed to be guaranteed. However, contrary to usual portfolio insurance, the initial exposure to the risky asset is significantly higher than the portfolio value (200% for example), such that a leveraged portfolio is obtained. Then a maximal bound equal to the initial exposure, 200% for example, is set on the risky asset exposure. This bound named the leverage is denoted by $L$. Therefore, the exposure is now given by:

$$e_k = \text{Min} \left[ m \times (V_k - \xi V_0); LV_0 \right]$$

with

$$L = m(1 - \xi).$$
In what follows, we denote by LCPPI the Leveraged CPPIs corresponding to $\xi = 50\%$ and by LCPPI 1 the Leveraged CPPIs corresponding to $\xi = 75\%$. If we fix the leverage $L$ at the level 2, we get:

1) $m = 4, \xi = 1/2$. This corresponds to LCPPI fund, in the next numerical comparison.

2) $m = 8, \xi = 3/4$. This corresponds to LCPPI 1 fund.

Note that, for the Leverage ETF strategy, the exposure is proportional to the portfolio value at any time. The proportion $L$ corresponds to the leverage as soon as we set $L$ higher than 1; for the Leveraged CPPI (LCPPI), the floor is proportional to the initial portfolio value. Thus, under condition $L = m(1 - \xi)$, the LETF and the LCPPI are identical on the first subperiod. They differ from the second subperiod since for the LETF the amount is proportional to portfolio value at time 1 ($e_1 = LV_1$) while for the LCPPI the amount is still partly based on the portfolio value at time 0 ($e_1 = \text{Min} \{m \times (V_1 - \xi V_0); LV_1\}$).

We illustrate all these features in the following example which considers three baseline scenarios: an upward trending market (+5% per day), a downward trending market (–5% per day) and a volatile market where a +5% increase is followed by a –5% decrease. While unrealistic, this data set is chosen for expositional purposes. However, note that the comments made from this example do not depend on these specific parameter values. For the first two scenarios, the volatility of the daily return is equal to zero, whereas for the last scenario the daily volatility is 5%. The first column contains the underlying index price, the next two columns show the leveraged (2x) ETF and the fund whose performance is twice that of the index over the whole period (static leverage). Columns (4) and (7) show a leveraged CPPI (LCPPI) portfolio which is designed to have maximum exposure to the risky asset of 200% of the total portfolio value. We also report the value of the exposure (columns (6) and (9)) if it had not been bounded above. The portfolios differ in the parameters used in the setting of the portfolio insurance strategy: LCPPI (resp. LCPPI 1) has a constant floor of

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10. For expositional simplicity and w.l.o.g., we assume here that interest rate is equal to zero.

11. As long as we consider an infinite sequence of returns.
50% (resp. 75%) of the portfolio value at inception and a multiple of 4 (resp. 8). The initial exposure to the risky asset of these two portfolios is set at 200%. Then a maximal bound of 200% is set on the risky asset exposure. However, note that these funds can reach less than 200% risky asset exposure during their lifetime. The difference between these two funds is that the one with the multiple $m$ of 8 is more sensitive to the fluctuations of the risky asset (its payoff is more convex in the value of the underlying index\footnote{See below Section (3.1) for a formal statement of this property.}) than the one with $m = 4$. Note also that the risk that this fund falls below the floor is higher. Table 1 reports data for this example. We compare first the performance of Leveraged ETF and Static Leveraged funds. In a trending market (upward or downward), the Leveraged ETF takes advantage of the daily compounding of the leveraged index return and exhibits a higher three-day return (33.10% in an upward market and –27.10% in a downward market) than the return of the static leveraged fund (31.53% and –28.52%).

Table 1. – Effect of Return Volatility and Compounding on $2 \times$ leveraged and LCPII Funds

|       | Daily Static Exposure | Exposure Exposure Exposure Exposure |
|-------|-----------------------|-----------------------|-----------------------|-----------------------|
|       | S Leveraged           | Leverage             | Leverage             | LCPII bounded Exposure Exposure |
|       | (1)                   | (2)                   | (3)                   | (4)                   | (5)                   | (6)                   | (7)                   | (8)                   | (9)                   |
| Upward trending market |                      |                       |                       |                       |                       |                       |                       |                       |                       |
| 100,00 | 100,00                | 100,00                | 100,00                | 100,00                | 200%                  | 200%                  | 100,00                | 200%                  | 200%                  |
| 105,00 | 110,00                | 110,00                | 100,00                | 100,00                | 200%                  | 218%                  | 110,00                | 200%                  | 255%                  |
| 110,25 | 121,00                | 121,00                | 100,00                | 100,00                | 200%                  | 235%                  | 121,00                | 200%                  | 304%                  |
| 115,76 | 133,10                | 131,53                | 133,10                | 133,10                | 200%                  | 250%                  | 133,10                | 200%                  | 349%                  |
| Downward trending market |                      |                       |                       |                       |                       |                       |                       |                       |                       |
| 100,00 | 100,00                | 100,00                | 100,00                | 100,00                | 200%                  | 200%                  | 100,00                | 200%                  | 200%                  |
| 95,00  | 90,00                 | 90,00                 | 90,00                 | 90,00                 | 200%                  | 178%                  | 90,00                 | 200%                  | 133%                  |
| 90,25  | 81,00                 | 82,00                 | 82,00                 | 84,00                 | 156%                  | 156%                  | 84,00                 | 86%                   | 86%                   |
| 85,74  | 72,90                 | 71,48                 | 75,60                 | 80,40                 | 135%                  | 135%                  | 80,40                 | 54%                   | 54%                   |
| Volatile Market |                      |                       |                       |                       |                       |                       |                       |                       |                       |
| 100,00 | 100,00                | 100,00                | 100,00                | 100,00                | 200%                  | 200%                  | 100,00                | 200%                  | 200%                  |
| 105,00 | 110,00                | 110,00                | 110,00                | 110,00                | 200%                  | 218%                  | 110,00                | 200%                  | 255%                  |
| 99,75  | 99,00                 | 99,00                 | 99,00                 | 99,00                 | 198%                  | 198%                  | 99,00                 | 194%                  | 194%                  |
| 104,74 | 108,90                | 109,48                | 108,80                | 108,60                | 200%                  | 216%                  | 108,60                | 200%                  | 248%                  |
Thus, in both cases and as we would expect, the leveraged ETF performs better than the static leveraged fund. Note also that, in both cases, the return volatility of the underlying index is nil. In a volatile market situation, the leveraged ETF return may be less than the return of the static leveraged fund, whether the underlying index experiences an increase (as in Table 1) or a decrease. The longer the buy-and-hold period, the stronger this effect, as we will see below. Volatility in financial markets has reached a very high level since the fall of 2008, which explains why investors may have very bad experiences with such funds. In an upward trending market, both LCPPI and LCPPI 1 funds have the same performance as the Daily Leveraged fund due to the bound on their exposure. But, in a downward trending market, these two funds take advantage of their trend-following feature by gradually cutting their exposure to the risky asset. Thus, they dampen the effect of the fall and exhibit better performance than the Daily Leveraged fund. LCPPI 1, with a higher multiple \((m = 8)\) than LCPPI \((m = 4)\), amplifies this effect. In the end, LCPPI strategies behaves rather poorly in volatile markets without trends.\(^\text{13}\)

3. **THE LEVERAGED PORTFOLIO VALUE**

In this section, first we recall that the leveraged portfolio strategy belongs to the class of the constant allocation portfolio strategies. Next, we establish under which assumptions a leveraged portfolio becomes a CPPI type portfolio in a continuous-time framework.

3.1. **The value of the leveraged and bear (i.e. inverse leveraged) strategy as a constant allocation (constant-mix) portfolio strategy**

A constant allocation (also named “constant mix”) portfolio strategy with two asset classes is a dynamically rebalanced strategy that aims to maintain exposures to both asset classes proportional to the portfolio value at any time.

\(^{13}\) The sensitivity to the volatility (the Vega) is negative for CPPI portfolios as shown in Bookstaber and Langsam (2001) and Bertrand and Prigent (2005).
3.1.1. The financial market

We adopt a simple continuous-time model where the stock index price\textsuperscript{14} dynamics is given by the following stochastic process:

\[ dS_t = S_t [\mu_t \, dt + \sigma_t \, dW_t], \]  
(11)

which implies:

\[ S_t = S_0 \exp \left[ \int_0^t (\mu_s - \frac{1}{2} \sigma_s^2) ds + \int_0^t \sigma_s \, dW_s \right], \]  
(12)

where \((W_t)_t\) is a standard Brownian motion with respect to a given filtration \((\mathcal{F}_t)_t\), \(S_0\) is the initial stock index price and \(\mu(.)\) and \(\sigma(.)\) are respectively the drift and volatility functions of the stock index price. These two processes are assumed to be adapted to the filtration \((\mathcal{F}_t)_t\) and satisfy usual assumptions. In this framework, we can consider, for instance, stochastic volatility models, as in Hull and White (1987) and Heston (1993). Note that when \(\mu(.)\) and \(\sigma(.)\) are constant, we recover the geometric Brownian motion (GBM hereafter) as a particular case.

The riskless asset price at time \(t\) is denoted by \(B_t\):

\[ B_t = B_0 \exp[rt], \]  
(13)

where \(r\) is the instantaneous riskless interest rate, which is assumed to be constant.

3.1.2. Value process

The time period considered is \([0,T]\) and the strategies are self-financing. We consider the constant allocation portfolio strategy in which the constant proportion \(L\) of initial wealth \(V_0\) is invested in the risky asset while the remainder proportion \((1 - L)\) is allocated to the riskless asset. Thus, the initial portfolio value is given by:

\[ V_0 = L V_0 + (1 - L) V_0, \]  
(14)

\[ = q^S_0 S_0 + q^B_0 B_0, \]  
(15)

\textsuperscript{14} In the context of the present article, the stock index is the ETF. Additionally, we assume that all price processes are dividend-adjusted.
where \( q^S_0 = \frac{LV_0}{S_0} \) and \( q^B_0 = \frac{(1 - L)V_0}{B_0} \) are the initial numbers of shares in the risky and riskless asset respectively. The portfolio is continuously adjusted so as to maintain the exposures to each asset proportional to the portfolio value.

Avellaneda and Zhang (2010) and Jarrow (2010) prove that the value of a constant allocation (constant-mix) self-financing portfolio strategy at any time is given by:

\[
V_t = V_0 \exp \left[ rt + \int_0^t \left( L (\mu_s - r) - L^2 \frac{\sigma_s^2}{2} \right) ds + L \int_0^t \sigma_s dW_s \right].
\]

(16)

This equation can be rewritten as:

\[
V_t = V_0 \left( \frac{S_t}{S_0} \right)^L \exp \left[ r (1 - L) t + \frac{1}{2} L (1 - L) \int_0^t \sigma_s^2 ds \right].
\]

(17)

From Relation (17), we can see that, depending on the value of \( L \), the portfolio value \( V_t \) is a convex or concave function of the stock index price \( S_t \):

- If \( 0 < L < 1 \), \( V_t \) is strictly concave.
- If \( L < 0 \) or \( L > 1 \), \( V_t \) is strictly convex.

Usually, constant-mix strategies are conceived for proportion \( L \) between zero and one. This is why these strategies are usually referred to as concave strategies. This strategy performs well under relatively flat but volatile market conditions. Moreover, it capitalizes on price reversals.

- If \( L < 0 \) or \( L > 1 \), \( V_t \) is strictly convex.

This case can be reduced without loss of generality to the case \( L > 1 \), the one that concerns us when considering leveraged funds. For instance, if \( L \) is set at 2, the portfolio holdings in the stock index are twice the portfolio’s value. The leveraged part is financed by borrowing \( V_t \) at the riskless rate.

15. See also Cheng and Madhavan (2009) for the GBM case.
16. The case, \( L < 0 \), a bear or inverse leveraged strategy, can be treated similarly.
Note also that with the index price dynamics given in (11), the LETF value is always above zero. But, as shown in Appendix A, this property is no longer true for the static case, for which continuous-time rebalancing is not allowed. Such difference between the discrete-time and continuous-time cases is well-known for the CPPI strategies.

3.1.3. Some statistical properties

Using Relation (16), we deduce:

**Proposition 1.** – If \( \mu(.) \) and \( \sigma(.) \) are deterministic, the leveraged ETF value has a Lognormal distribution:

\[
\frac{V_t}{V_0} \sim \text{LnN} \left( rt + L \int_0^t (\mu_s - r) ds - \frac{L^2}{2} \int_0^t \sigma_s^2 ds; L \sqrt{\int_0^t \sigma_s^2 ds} \right).
\]

**Remark 2.** – In the GBM case (\( \mu \) and \( \sigma \) constant), we have:

\[
\frac{V_t}{V_0} \sim \text{LnN} \left( (r + L (\mu - r) - L^2 \sigma^2 / 2) t; L \sigma \sqrt{t} \right).
\]

Figures 1 and 2 display the probability density function (pdf) and the cumulative distribution function (CDF) of the return on the leveraged ETF for different values of the leverage parameter \( L \) in the GBM case.

As shown in Figure 2, the higher the level \( L \), the higher the probability to get significant positive returns but also the higher the probability to get significant negative returns. For example, for \( L = 2 \) (resp. \( L = 5 \) ), the probability that the return is higher than 50% is equal to about 20% (resp. 30%) and the probability to get a return value smaller than −50% is equal to about 0.05% (resp. 30%).

Using results about Lognormal distributions, we get:

**Proposition 3.** – If \( \mu(.) \) and \( \sigma(.) \) are deterministic, the first four moments of the leveraged ETF value are given by:

\[
\begin{align*}
\mathbb{E}\left[ \frac{V_t}{V_0} - \frac{V_0}{V_0} \right] &= \exp \left( rt + L \int_0^t (\mu_s - r) ds \right) - 1, \\
\text{Var}\left[ \frac{V_t}{V_0} - \frac{V_0}{V_0} \right] &= \exp \left( 2rt + 2L \int_0^t (\mu_s - r) ds \right) \left[ \exp \left( L^2 \int_0^t \sigma_s^2 ds \right) - 1 \right], \\
\text{Skew}\left[ \frac{V_t}{V_0} - \frac{V_0}{V_0} \right] &= \left[ 2 + \exp \left( L^2 \int_0^t \sigma_s^2 ds \right) \right] \sqrt{\exp \left( L^2 \int_0^t \sigma_s^2 ds \right) - 1}, \\
\text{Kurt}\left[ \frac{V_t}{V_0} - \frac{V_0}{V_0} \right] &= \exp \left( 4L^2 \int_0^t \sigma_s^2 ds \right) + 2\exp \left( 3L^2 \int_0^t \sigma_s^2 ds \right) + 3\exp \left( 2L^2 \int_0^t \sigma_s^2 ds \right) - 3.
\end{align*}
\]
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Figure 1. – Cdf of \( \left( \frac{V_t}{V_0} - 1 \right) \) for various \( L \) \((t = 1)\)

Figure 2. – Cdf of \( \left( \frac{V_t}{V_0} - 1 \right) \) for various \( L \) \((t = 1)\)
Note that, if for any \( s, \mu_s > r \), both expectation and variance are increasing functions of the leverage parameter \( L \). This implies that no such leveraged ETF dominates another with respect to the mean-variance criterion. The (relative) skewness and kurtosis are also increasing with respect to the leverage parameter \( L \).

In Table 2, the first four moments of the return of these leveraged ETFs are displayed. They illustrate the strong effect of leverage on these statistics. The following values are used for the financial market parameters: \( \mu = 8\% \), \( r = 3\% \) and \( \sigma = 20\% \).

<table>
<thead>
<tr>
<th>( L )</th>
<th>Median (%)</th>
<th>Expectation (%)</th>
<th>Volatility (%)</th>
<th>Skewness</th>
<th>Kurtosis</th>
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<td>1</td>
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<tr>
<td>5</td>
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<td>32.31</td>
<td>173.4</td>
<td>6.185</td>
<td>113.9</td>
</tr>
</tbody>
</table>

From Equation (16), if \( \mu(.) \) and \( \sigma(.) \) are deterministic, the expected continuous time growth rate over the period \([0,t]\) is equal to

\[
(r + \frac{1}{t} \int_0^t (\mu_s - r) \, ds - \frac{1}{2} \frac{1}{t} \int_0^t \sigma_s^2 \, ds) \quad \text{(see Giese (2010a) when \( \mu(.) \) and \( \sigma(.) \) are constant).}
\]

Apart from time, it has three components\(^{17}\):

- The riskless rate, \( r \);
- The temporal mean of the leveraged excess return of \( S \) over \( r \), \( \frac{1}{t} \int_0^t (\mu_s - r) \, ds \);
- A variance term, \( -\frac{1}{2} L(L - 1) \frac{1}{t} \int_0^t \sigma_s^2 \, ds \), that reduces the expected Logreturn irrespective of the sign of \( L \). The term is proportional to the temporal mean of the realized variance \( \frac{1}{t} \int_0^t \sigma_s^2 \, ds \) of the ETF.

\(^{17}\) Alternatively, the first two components can be organized in the following way:

- Leveraged return on the index, \( L \frac{1}{t} \int_0^t \mu_s \, ds \),
- Financing cost, \( r(1 - L) < 0 \) for \( L > 1 \).

The volatility correction is the same.
The variance correction term can even be such that the expected logreturn becomes negative. It is maximized for the value of the leverage given by:

\[
L^* = \frac{\int_0^t (\mu_s - r) \, ds}{\int_0^t \sigma_s^2 \, ds},
\]

where it reaches the value \( r + \frac{1}{2t} \left( \frac{\int_0^t (\mu_s - r) \, ds}{\int_0^t \sigma_s^2 \, ds} \right)^2 \). Relation (18) shows that the optimal leverage is stochastic and corresponds to a Sharpe-type ratio. For the GBM case, this optimal level is indeed constant. For instance, for previous financial parameter values, we obtain: \( L^* = 1.25 \).

It is possible to give another but equivalent interpretation of the value process, based on relation (17). The growth return of the LETF over the period \([0, t]\) is equal to the underlying index return compounded \( L \) times reduced by the interest paid on \([0, t]\), \( \exp[r(1 - L)t] \), and by the effect of the ETF’s volatility on \([0, t]\), \( \exp[\frac{1}{2} L (1 - L) \int_0^t \sigma_s^2 \, ds] \).

As already shown by Bertrand and Prigent (2003, 2005) for the CPPI case, the portfolio value given by Relation (17) is inversely related to the volatility of the risky asset. In other words, the Vega of the leveraged ETF is negative. Note that it is the realized volatility that matters here, and not the expected (or implied) volatility, as is the case for option prices.

### 3.1.4. A troubling feature about LETF and risky asset returns

In what follows, we examine how the leveraged fund value varies according to the risky asset fluctuations, for the GBM case.

From Relation (17), we deduce:

\[
\ln \left( \frac{V_t}{V_0} \right) = L \ln \left( \frac{S_t}{S_0} \right) + \left[ r(1 - L)t + \frac{1}{2} L (1 - L) \int_0^t \sigma_s^2 \, ds \right].
\]

(19)

Therefore, in continuous-time, the portfolio logreturn is equal to the risky asset logreturn multiplied by the leverage parameter \( L \) to which a correction term is added (this is due to the compound rates and involves the volatility term). The impact of this latter term has to be empha-
sized. Indeed, the portfolio return is an increasing function of the risky asset return but the correction term is negative for values of the leverage coefficient $L$ higher than one.

The first implication is that the logreturn of the fund $V$ is always smaller than $L$ times that of the asset $S$. However, it is not always true for the comparison between the arithmetical return on the fund value $V$ and $L$ times that of the asset value $S$.

**Proposition 4.** – The probability $P_1$ that the arithmetical return on the fund value $V$ is less than $L$ times that of the asset value $S$ is given by:

$$P_1 = \Phi \left( \frac{\ln \left[ x_{a,L}^{(1)} \right] - \left( (\mu - \frac{1}{2} \sigma^2) t \right)}{\sigma \sqrt{t}} \right) - \Phi \left( \frac{\ln \left[ x_{a,L}^{(2)} \right] - \left( (\mu - \frac{1}{2} \sigma^2) t \right)}{\sigma \sqrt{t}} \right),$$  (20)

where $\Phi$ denotes the cumulative distribution function (cdf) of the standard Gaussian distribution and where $x_{a,L}^{(1)}$ and $x_{a,L}^{(2)}$ are the two solutions of equation: $x e^a - Lx + (L - 1) = 0$, with $a = r (1 - L)t + \frac{1}{2} L (1 - L) \sigma^2 t$.

**Proof.** – Denote $Y = \frac{V_t}{V_0}$, $X = \frac{S_t}{S_0}$. Then, from (17), we get:

$$Y = X^L \exp[a].$$  (21)

Note that $a < 0$ as soon as $L > 1$. We have to examine the value $\mathbb{P}[Y - 1 < L(X - 1)]$. Condition $Y - 1 < L(X - 1)$ is equivalent to:

$$X^L e^a - LX + (L - 1) < 0.$$

The function $f_{a,L}(x) = x^L e^a - Lx + (L - 1)$ has a minimum at $\exp[-a/(L - 1)]$ on $\mathbb{R}^+$ with $f_{a,L}(\exp[-a/(L - 1)]) < 0$. Since $f_{a,L}(0) = L - 1 > 0$ and $\lim_{x \to +\infty} f_{a,L}(x) = +\infty$, there exists
exactly two non negative numbers $x_{a,L}^{(1)}$ and $x_{a,L}^{(2)}$ such that:

For $x > 0$, $x^L e^a - Lx + (L - 1) < 0 \iff x_{a,L}^{(1)} < x < x_{a,L}^{(2)}$.

Using the equality $\ln|X| = (\mu - \frac{1}{2}\sigma^2) t + \sigma \sqrt{t} Z$ where $Z$ has a standard Gaussian distribution, we prove Relation (20).

\[ P_1 = \Phi \left( \frac{\ln \left[ \frac{1 + \sqrt{1 - e^{-e^{a}\sqrt{1 - \mu}}}}{e^{a}} \right] - (\mu - \frac{1}{2}\sigma^2) t}{\sigma \sqrt{t}} \right) - \Phi \left( \frac{\ln \left[ \frac{1 - \sqrt{1 - e^{-e^{a}\sqrt{1 - \mu}}}}{e^{a}} \right] - (\mu - \frac{1}{2}\sigma^2) t}{\sigma \sqrt{t}} \right), \]  

(22)

**Proof.** – It is sufficient to determine the explicit solutions $x_{a,2}^{(1)}$ and $x_{a,2}^{(2)}$ of equation: $x^2 e^a - 2x + 1 = 0$. We have:

$x_{a,2}^{(1)} = \frac{1 - \sqrt{1 - e^a}}{e^a}$ and $x_{a,2}^{(2)} = \frac{1 + \sqrt{1 - e^a}}{e^a}$,

with $a = -[r + \sigma^2] t$. \[ \square \]

For example, for $\mu = 8\%$, $\sigma = 20\%$, $r = 3\%$, $t = 0.5$, $L = 2$, we get $P_1 \simeq 92\%$. Figure 3 shows that the probability $P_1$ is decreasing with respect to the volatility $\sigma$ but is less sensitive to the expected return $\mu$ and time.

The second and main implication is that the stock index can experience an increase while at the same time the leveraged fund decreases. This is an event that is probably difficult to accept for an individual who invests on such fund.
Proposition 6. The probability $P_2$ that the stock index increases while the leveraged fund decreases, denoted by $P_2$, is given by:

$$P_2 = \Phi\left(-\frac{a}{L} - \frac{1}{2} \frac{\mu - \frac{1}{2} \sigma^2}{\sigma \sqrt{t}} t\right) - \Phi\left(-\frac{\mu - \frac{1}{2} \sigma^2}{\sigma \sqrt{t}} t\right). \quad (23)$$

Note that, by construction, we have always $P_1 \geq P_2$.

Proof. With the same notations as in previous proof, we are looking for the value $\mathbb{P}[X > 1] \cap (Y < 1)]$. We observe that

$$Y < 1 \Leftrightarrow X^L \exp[a] < 1, \quad (24)$$

$$\Leftrightarrow X < \exp[-a/L](> 1). \quad (25)$$

Thus, we must evaluate the probability that $1 < X < \exp[-a/L]$, which is equivalent to: $0 < \text{Ln}(X) < -a/L$.

Therefore, we deduce:

$$\mathbb{P}[0 < \text{Ln}(X) < -a/L] = \Phi\left(-\frac{a}{L} - \frac{1}{2} \frac{\mu - \frac{1}{2} \sigma^2}{\sigma \sqrt{t}} t\right) - \Phi\left(-\frac{\mu - \frac{1}{2} \sigma^2}{\sigma \sqrt{t}} t\right).$$

□
We can gain some intuition on this phenomenon by recalling that $V$ is solution of the following stochastic differential equation:

$$dV_t = V_t \left[(1 - L)dB_t/B_t + LdS_t/S_t\right], \text{ with } L > 1.$$  

Thus, even if $S$ increases (but not sufficiently), the short position on the riskless asset $B$ can induce a decrease in LETF value.\(^\text{18}\)

We can also look at equation (17), which provides the LETF (compound) value at any time of the management period. On the one hand and in the event that the return on the risky asset is positive, the first term in the right-hand side of equation (17) tells us that the return on the fund is greater than that of the risky asset because of the leverage effect. On the other hand, the second term is less than one. Thus, the product of these two terms might be less than the return of the risky asset. This is even more likely that the price $S_t$ is low, that the volatility, the leverage, the riskless rate and the time are high. A situation where the terminal price may be low and the volatility is high, is typically what financial markets have experienced recently.

In what follows, we examine several sensitivities of the probability $P_2$ that the stock index increases while the leveraged fund decreases.

**Proposition 7.** - **Sensitivities of the probability $P_2$:**

i) The derivative of the probability $P_2$ with respect to the volatility $\sigma$ is given by:

$$\frac{\partial P_2}{\partial \sigma} = \left(\frac{(L - 1)\sqrt{t} \left(\frac{1}{2}L\sigma^2 - r\right)}{L\sigma^2} + \frac{\mu\sqrt{t}}{\sigma^2} + \frac{\sqrt{t}}{2}\right) \Phi'\left(-\frac{a/L - \left(\mu - \frac{1}{2}\sigma^2\right)t}{\sigma\sqrt{t}}\right) - \left(\frac{\mu\sqrt{t}}{\sigma^2} + \frac{\sqrt{t}}{2}\right) \Phi'\left(-\frac{\left(\mu - \frac{1}{2}\sigma^2\right)t}{\sigma\sqrt{t}}\right).$$

ii) The derivative of the probability $P_2$ with respect to the instantaneous expected return $\mu$ is given by:

18. In discrete-time, this probability corresponds to

$$\mathbb{P}\left[\frac{\Delta V_t}{V_{t-1}} < 0 \cap \frac{\Delta S_t}{S_{t-1}} > 0\right] = \mathbb{P}\left[0 < \frac{\Delta S_t}{S_{t-1}} < \frac{L - 1}{L} \frac{\Delta B_t}{B_{t-1}}\right].$$
\[
\frac{\partial P_2}{\partial \mu} = -\frac{1}{\sigma \sqrt{t}} \left[ \Phi'\left(\frac{-a/L - (\mu - \frac{1}{2}\sigma^2) t}{\sigma \sqrt{t}}\right) - \Phi'\left(\frac{- (\mu - \frac{1}{2}\sigma^2) t}{\sigma \sqrt{t}}\right) \right].
\]  
(29)

iii) The derivative of the probability \(P_2\) with respect to the leverage \(L\) is given by:

\[
\frac{\partial P_2}{\partial L} = \left(\frac{r + \frac{1}{2} L^2}{\sigma^2 \sqrt{t}}\right) \Phi'\left(\frac{-a/L - (\mu - \frac{1}{2}\sigma^2) t}{\sigma \sqrt{t}}\right).
\]  
(30)

iv) The derivative of the probability \(P_2\) with respect to the interest rate \(r\) is given by:

\[
\frac{\partial P_2}{\partial r} = \left(\frac{(L - 1) \sqrt{t}}{\sigma L}\right) \Phi'\left(\frac{-a/L - (\mu - \frac{1}{2}\sigma^2) t}{\sigma \sqrt{t}}\right).
\]  
(31)

v) The derivative of the probability \(P_2\) with respect to the time to maturity \(t\) is given by:

\[
\frac{1}{2\sigma \sqrt{t}} \left( \left(1 - \frac{1}{L}\right) r + \frac{1}{2} L \sigma^2 - \mu \right) \Phi'\left(\frac{-a/L - (\mu - \frac{1}{2}\sigma^2) t}{\sigma \sqrt{t}}\right) + \left(\mu - \frac{1}{2}\sigma^2\right) \frac{2}{\sigma \sqrt{t}} \Phi'\left(\frac{- (\mu - \frac{1}{2}\sigma^2) t}{\sigma \sqrt{t}}\right)
\]

Proof. – See Appendix B.  

Applying results of Proposition 7, we can prove the following properties:

**Corollary 8.** – Under specific assumptions, the derivative \(\frac{\partial P_2}{\partial \sigma}\) is equal to 0 for a particular value of \(\sigma_0\) and the derivative \(\frac{\partial P_2}{\partial \mu}\) is equal to 0 for some standard value of \(\mu\). In that case, the probability \(P_2\) is an increasing then decreasing function of the instantaneous expected return \(\mu\). The probability \(P_2\) is increasing with respect to the time to maturity \(t\). The derivatives \(\frac{\partial P_2}{\partial L}\) and \(\frac{\partial P_2}{\partial r}\) are always non negative.

19. See Appendix B and next numerical example.
20. In the example, \(\mu \simeq 3.8\%\).
Proof. – See Appendix B.

To illustrate these properties, we consider the following parameter values to assess the magnitude of the probability $P_2$: $\mu = 8\%$, $\sigma = 20\%$, $r = 3\%$, $t = 0.5$, $L = 2$.

Therefore, the probability $P_2$ is equal to 4.9\%, and if the management period is set at one year, this probability becomes 6.82\%. This feature is not negligible and makes a difference to investors. Figure 4 displays the different sensitivities of the probability $P_2$.

As mentioned in Corollary 8, the behavior of the probability $P_2$ as a function of the volatility of the stock index is rather complicated, since it also depends on the value of the other parameters. For small values of $t$ (typically less than one year), it is first increasing, then reaches a local maximum, then becomes decreasing again to reach a local minimum before finally becoming increasing again. Nevertheless, for values of the volatility between 5\% and 40\%, the probability is between 5.57\% and 6.67\%, meaning that the sensitivity is small. The probability $P_2$ is increasing in the leverage because the ratio $(-a/L)$ is also increasing in the leverage. Thus, we are computing a probability of the same random variable on a wider interval. Moreover, it shows a high degree of sensitivity, since it rises from 4.5\% to 14.5\% when the leverage goes from 2 to 5. As a function of the expected return of the underlying index, the probability $P_2$ is first slightly increasing and then decreasing. Note nevertheless that the magnitude of the effect is small. For example, when the expected return of the risky asset is between 5\% and 15\%, the probability is not very sensitive to it and lies between 4.93\% and 4.55\%. This arises because expected return has an opposite effect on both parts of the right-hand side of expression (23). The probability $P_2$ is an increasing function of the riskless interest rate (almost linear). The effect of the riskless rate on the probability is stronger than that of the expected return. For $r$ equal to 2\%, the probability is equal to 4.18\% and for $r$ equal to 6\%, it is equal to 6.99\%. This is because interest rate enters Formula (23) only in the first cumulative distribution function. Finally, for this numerical example, the probability $P_2$ is an increasing and concave function of time.

We can gain more insight into the behavior of the probability $P_2$ as a function of the volatility by considering simultaneously the effect of another variable. In the following two figures, we draw 3D plots where
the expected return and the time are introduced along with the volatility. On this figure, we can see that the local maximum is only obtained for very small level of the volatility independently of the value taken by the expected return. Moreover, the value of this local maximum is strongly decreasing with the value of the expected return.
3.2. CPPI equivalence

In this section, we establish that, in continuous-time, a CPPI portfolio with a floor proportional to the value of the portfolio itself is a constant allocation portfolio strategy. First, recall that usual CPPI strategy provides portfolio insurance and is designed to give the investor the ability to limit downside risk while allowing some participation in upside markets. Such method allows investors to recover, at maturity, a given percentage of their initial capital, in particular in falling markets. The CPPI method consists of managing a dynamic portfolio so that its value is above a floor $F$ at any time $t$. The value of the floor gives the dynamically insured amount. It is assumed to evolve according to:

$$dF_t = F_t r dt$$

Obviously, the initial floor $F_0$ is less than the initial portfolio value $V_0^{CPP}\text{P}$. This difference $V_0^{CPP}\text{P} - F_0$ is called the cushion, denoted by $C_0$. Its value $C_t$ at any time $t$ in $[0, T]$ is given by:

$$C_t = V_t^{CPP}\text{P} - F_t$$

Denote by $e_t$ the exposure, which is the total amount invested in the risky asset. The standard CPPI method consists in letting $e_t = mC_t$
where $m$ is a constant called the multiple. The interesting case is when $m > 1$, that is, when the payoff function is convex. Thus, here the constant proportion is between the amount invested in the risky asset and the cushion, whereas in constant-mix strategies, this constant proportion is maintained between the amount invested in the risky asset and the portfolio value.

The value of this portfolio $V_{t}^{CPPI}$ at any time $t$ in the period $[0,T]$ is given by:\(^{21}\):

$$V_{t}^{CPPI} = F_{0}e^{rt} + C_{0}\exp[rt + m\int_{0}^{t}(\mu_{s} - r - \frac{m\sigma_{s}^{2}}{2})ds + m\int_{0}^{t}\sigma_{s}dW_{s}] = F_{0}e^{rt} + C_{0}\left(\frac{S_{t}}{S_{0}}\right)^{m}\exp[r(1-m)t - (m^{2} - m)\int_{0}^{t}\frac{\sigma_{s}^{2}}{2}ds].$$

Thus, the CPPI method is parametrized by the initial floor $F_{0}$ and the multiple $m$.

We now study the link between CPPI portfolio and constant-mix portfolio strategy. It is usually believed that a CPPI with a zero floor and $0 < m < 1$ is a necessary and sufficient condition for this strategy to be a constant mix strategy.\(^{22}\) Nevertheless, we show in what follows that, at least formally, it is not necessary that the floor is equal to zero. More precisely, in the following proposition we discuss the conditions under which these two methods are equivalent.

**Proposition 9.** – A CPPI portfolio with a multiple $m$ and with a stochastic floor $F$ proportional to the portfolio value at any time $t$ in the period $[0,T]$ (proportion equal to $\xi$) is a constant-mix portfolio strategy with leverage coefficient $L$ if and only if we have:

$$L = m(1 - \xi).$$

**Proof.** – A floor proportional to the portfolio value at any time $t$ in the period $[0,T]$ is written:

$$F_{t} = \xi V_{t}^{CPPI}, \quad 0 < \xi < 1.$$  \(^{32}\)

21. This formula was proved by Black and Perold (1987) and has been extended to the stochastic volatility case by Bertand and Prigent (2003).

22. See Escobar et al. (2009).
Then, the risky exposure is written:

\[ e_t = m \left( V_{t}^{CPPI} - F_t \right) = m(1 - \xi)V_t^{CPPI}. \]  

(34)

The evolution at time \( t \) of the CPPI portfolio value is given by

\[ dV_t^{CPPI} = (1 - m(1 - \xi))V_t^{CPPI}\frac{dB_t}{B_t} + m(1 - \xi)V_t^{CPPI}\frac{dS_t}{S_t}, \]

which implies that:

\[ V_t^{CPPI} = V_0 \left( \frac{S_t}{S_0} \right)^{m(1-\xi)} \exp[r(1 - m(1 - \xi))t] \]

\[ + \frac{1}{2}m(1 - \xi)(1 - m(1 - \xi)) \int_0^t \sigma_s^2 ds. \]  

(35)

It can be checked that setting \( L = m(1 - \xi) \) leads Equation (26) which describes the evolution of the constant allocation portfolio value. □

Leveraged CPPI portfolios at inception are obtained when \( m > 1/(1 - \xi) \) (i.e. \( e_0 > V_0^{CPPI} \)).

Thus, the critical feature that makes a constant proportion strategy becoming a constant allocation strategy is that the floor is proportional to the portfolio value at any time. This means that, contrary to the usual CPPI, there is no longer any downside protection because the floor is now stochastic and can therefore reach zero\(^{23}\).

4. COMPARISON OF LEVERAGED ETF AND LEVERAGED CPPI PORTFOLIO

In this section, we first provide a detailed analysis of the value process of the leveraged CPPI portfolio when, contrary to previous sec-

\(^{23}\) We could have set the value of the floor such that: \( F_t = \xi V_t^{CPPI} \equiv F_0, \)

\( 0 < \xi < 1. \) In this case, we would obtain a constant-mix-type strategy in which the part of the wealth invested in the risky asset is no longer proportional to the whole wealth at each time, but rather to \( (V_t^{CPPI} - F_0). \)
tion, the floor is not always equal to a fixed proportion of the portfolio value but only equal to it at a finite number of dates. This case is illustrated by the SGAM Leveraged CAC 40 strategy. It corresponds to a maximum exposure to the risky asset of 200%. Then, we consider other parameter values for the multiple $m$ and the percentage $\xi$ with a two standard values of the leverage coefficient $L$: $L = 2$ or $L = 3$. We examine the respective cumulative distribution functions of their returns. We also introduce Kappa measures to compare their performances. Even if we assume that this strategy is rebalanced in continuous time, some features of this product prevent a fully explicit formula for the value of the leveraged CPPI portfolio from being obtained. This is because the floor is adjusted at the beginning of each month and then held constant during the month. We conduct a Monte Carlo simulation in discrete time in order to compare LETF and leveraged CPPI portfolio.

4.1. Leveraged CPPI

The Leveraged CPPI fund is very similar to the CPPI funds introduced in Section 1. For example the fund denoted LCPPI with parameter values $m = 4$ and $\xi = 50\%$ is designed to have maximum exposure to the risky asset of 200%. However, this fund may reach less than 200% risky asset exposure during its lifetime. In this respect, it differs from the standard LETF, for which the leverage is constant through time. It also has a monthly floor of 50% of the start-of-the-month portfolio value, which remains constant during the rest of the month. The value of the multiple at the inception of the fund is set at 4.6 and subsequently allowed to vary between 3.00 and 5.50. It is readjusted at the beginning of each quarter. This point is not addressed here because it is a tactical asset allocation decision. For expositional purposes, we consider a constant multiple $m$ ($m = 4$ for this first example) with an exposure that is assumed to be upper bounded: it is capped with leverage coefficient $L = 2$ (see proposition below).

Although no insurance is provided for the whole period of management and, as a consequence, the entire amount invested may be lost, insurance is obtained within each month.
Consider the sequence \((t_n)_{n}\) of times at which the floor is determined by the relation\(^{24}\):

\[ F_{t_n} = \xi V_{t_n} \]  
with \(0 < \xi < 1\). \hfill (36)

Usually, \(t_n\) is the first day of the \(n\)th month of the portfolio management period. Assume that portfolio is rebalanced in discrete time during each period \([t_n, t_{n+1}]\). Let \(t_{n,l} = t_n + l h_n\) the rebalancing times during \([t_n, t_{n+1}]\), where \(l \in \{0, \ldots, \frac{t_{n+1} - t_n}{h_n}\}\) and \(h_n\) denotes the rebalancing frequency (typically, \(h_n\) corresponds to one day).

Then, taking account of all previous assumptions about leveraged CPPI leads to:

**Proposition 10.** – The exposure of the leveraged CPPI is given by:

\[ e_{t_n,l} = \max \left\{ \min \left[ m \times (V_{t_n,l} - F_{t_n}); L V_{t_n,l} \right]; 0 \right\}, \]  
where \(m\) is the multiple and \(L\) denotes the leverage coefficient.

The leveraged CPPI portfolio value satisfies:

\[ V_{t_n,l} = \max \left[ e_{t_n,l-1} \times \left( \frac{S_{t_n,l}}{S_{t_n,l-1}} \right) + (V_{t_n,l-1} - e_{t_n,l-1}) \times (1 + rh_n); F_{t_n} \right], \]  
where the parameter \(L\) is the maximum exposure allowed for the fund\(^{25}\).

Relation (38) does not allow an explicit solution but can be numerically illustrated by using Monte Carlo simulations (see next subsection). However, under some additional mild assumptions, we can provide a quasi-explicit solution (see Appendix C).

\(^{24}\) The parameter \(\xi = 50\%\) for instance.

\(^{25}\) In the case of the SGAM product, it was set at 200\%. 
4.2. Simulations and comparisons of Leveraged ETF and Leveraged CPPI Portfolios

In this section, we implement Monte Carlo simulations to compare both strategies with a management period of one year. A discretized geometric Brownian motion is simulated on a daily basis over a period of 240 days. We use the same values of market parameters as previously and 1,000,000 paths for the underlying stock index are drawn. Then, a 2x LETF and a 3x LETF portfolio strategy are simulated. We consider three Leveraged CPPI strategies described in Table 3:

Table 3. – Leveraged CPPI Strategies

<table>
<thead>
<tr>
<th></th>
<th>$L = 2$</th>
<th>$L = 3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\xi$</td>
<td>$m$</td>
<td>$\xi$</td>
</tr>
<tr>
<td>LCPPI</td>
<td>50%</td>
<td>4</td>
</tr>
<tr>
<td>LCPPI 1</td>
<td>75%</td>
<td>8</td>
</tr>
<tr>
<td>LCPPI 2</td>
<td>90%</td>
<td>20</td>
</tr>
</tbody>
</table>

All three strategies are designed to provide an initial exposure to the risky asset of 200% (resp. 300%) when $L = 2$ (resp. $L = 3$). Although the last case, LCPPI 2, is unrealistic, the value of the multiple being much too high, especially for a stock index, it is used for expositional purposes. These strategies are also simulated along the 1,000,000 paths of the stock index.

Table 4 show the first four moments of the return corresponding to the four strategies for a leverage of 2 and 3.

Table 4. – First four Moments of Returns of LETF and Leveraged CPPI Strategies

<table>
<thead>
<tr>
<th>$L = 2$</th>
<th>$L = 3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>LETF</td>
<td>LCPPI</td>
</tr>
<tr>
<td>Expectation (%)</td>
<td>13.90</td>
</tr>
<tr>
<td>Volatility (%)</td>
<td>47.41</td>
</tr>
<tr>
<td>Skewness</td>
<td>1.31</td>
</tr>
</tbody>
</table>

26. Here a stylized year is assumed to contain twelve months of 20 days.
27. Notice that for $L = 3$, the case $m = 12$ is also unrealistic.
For a leverage of two, the first two portfolio strategies come out as very close. Thus, the LCPPI strategy is essentially equivalent to a 2x LETF with a little less expected return and a little more downside protection, as will be highlighted in Figure 6. The LCPPI 1 and LCPPI 2 portfolios exhibit less expected returns, less volatility, more positive asymmetry and more extreme risk than the first two strategies.

When considering a leverage of three, the difference between the first two portfolios is a little bit more significant.

Figure 6 represents the cumulative distribution function (CDF) of the four portfolio strategies for a leverage of 2. It highlights the downside protection that is provided by the Leveraged CPPI strategies. This effect is all the more marked as the values of the multiple $m$ and of the floor parameter $\xi$ are high.

For instance, the probability that the return on the LETF is less than $-20.64\%$ is $24.84\%$, whereas the same probability for the LCPPI 2 fund is $20.72\%$. The LETF curve crosses the LCPPI curve at a return level of $0.794\%$, the LCPPI 1 curve at a return level of $-1.51\%$ and the LCPPI 2 curve at a return level of $-2.900\%$.

**Figure 6.** CDF of the four portfolio strategies, $L = 2$. 
Figure 7 represents the CDF of the four portfolio strategies for a leverage of 3. It highlights the downside protection that is provided by the Leveraged CPPI strategies. This effect is all the more marked as the values of the multiple \( m \) and of the floor parameter \( \xi \) are high.

![Figure 7](image)

**Figure 7. – CDF of the four portfolio strategies, \( L = 3 \)**

For instance, the probability that the return on the LETF is less than -21.54\% is 35.074\%, whereas the same probability for the LCPPI 2 fund is 30.61\%. The LETF curve crosses the LCPPI curve at a return level of 6.280\%, the LCPPI 1 curve at a return level of 1.910\% and the LCPPI 2 curve at a return level of -2.941\%. It implies that the probability to bear any loss is always higher for the LETF than for LCPPI and LCPPI 1, while the probability to bear any loss beyond -3\% is higher for the LETF than for the LCPPI 2.

Another point needed to be addressed is the case where the value of the multiple of the LCPPI strategy with \( \xi = 50\% \) is such that the median returns on the LETF and on the LCPPI funds are equal. In our example, this value of the multiple denoted by \( m^\ast \) is approximately 4.5625 (resp. 6.65) for \( L = 2 \) (resp. \( L = 3 \)). We thus obtain the moments for the two strategies that appear in Table 5.
Thus for a 2x leverage, when the medians are set equals, the LCPPI slightly dominates in a median-variance sense (as well as in a mean-variance sense) the LETF. Even the skewness is better for the LCPPI, just the kurtosis is worse. For the 3x leverage case, the slight median-variance (as well as the mean-variance) dominance is reversed.

We can proceed in the same manner but with equalization of expected returns. Thus, we are led to consider the case where the value of the multiple of the LCPPI strategy with $\xi = 50\%$ is such that the expected returns on the LETF and on the LCPPI funds are equal. In our example, this value of the multiple denoted by $m^*$ is approximately 4.475 (resp. 7.4005) for $L = 2$ (resp. $L = 3$). We thus obtain the moments for the two strategies that appear in Table 6.

**Table 6. – First four Moments of LETF and LCPPI Strategy in the case of equal means**

<table>
<thead>
<tr>
<th></th>
<th>$L = 2$</th>
<th>$L = 3$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>LETF</td>
<td>LCPPI</td>
</tr>
<tr>
<td>Median (%)</td>
<td>5.172</td>
<td>5.150</td>
</tr>
<tr>
<td>Expectation (%)</td>
<td>13.88</td>
<td>13.88</td>
</tr>
<tr>
<td>Volatility (%)</td>
<td>47.36</td>
<td>47.20</td>
</tr>
<tr>
<td>Skewness</td>
<td>1.313</td>
<td>1.321</td>
</tr>
<tr>
<td>Kurtosis</td>
<td>6.21</td>
<td>6.25</td>
</tr>
</tbody>
</table>

In this case, the LCPPI strategy is almost equivalent to the LETF. Actually, it is slightly more dominant if we consider only the first three moments.
4.3. Omega and Kappa performance measures of Leveraged ETF and Leveraged CPPI Portfolios

In this section, we compare leveraged ETF and leveraged CPPI strategies by means of the Omega performance measure. Indeed, both strategies are not linear in the risky asset due to the leverage characteristic. This induces asymmetric returns and higher probability of loss. Thus, downside risk measures are required for such products. The Omega and the Kappa performance measures are based on such risk measures.28

The Omega performance measure has been introduced by Keating and Shadwick (2002) and Cascon et al. (2003) to take more account of the potential losses than standard performance measures such as the Sharpe ratio based only on the mean and the variance of the distribution of the returns. Omega measure involves the entire return distribution by separating the returns below and above a fixed loss threshold since Omega is equal to the probability weighted ratio of gains to losses relative to a return threshold \( q \):

\[
\Omega_X(q) = \frac{\int_q^b (1 - F(x)) \, dx}{\int_a^q F(x) \, dx},
\]

where \( F(\cdot) \) is the cumulative distribution function of the asset return \( X \) defined on the interval \((a, b)\), with respect to the probability distribution \( \mathbb{P} \) and \( q \) is the return threshold selected by the investor. Returns below the loss threshold \( q \) are viewed as losses and returns above as gains. For a fixed threshold, investor should always prefer the portfolio with the highest Omega value.

As proved by Kazemi et al. (2004), Omega can be written as:

\[
\Omega_X(q) = \frac{\mathbb{E}_\mathbb{P}[(X - q)^+]}{\mathbb{E}_\mathbb{P}[(q - X)^+]},
\]

Thus Omega can be considered as the ratio of the prices of a call option to a put option written on \( X \) with strike price \( q \) but both evaluated under the historical probability \( \mathbb{P} \).

---

28. We refer to Bertrand and Prigent (2011) for an in-depth comparison of standard portfolio insurance strategies with the Omega and the Kappa measures.
Kazemi et al. (2004) introduce the Sharpe Omega measure defined by:

\[
\text{Sharpe}_\Omega(q) = \frac{\mathbb{E}_P[X] - q}{\mathbb{E}_P[(q - X)^+]} = \Omega_X(q) - 1. \tag{39}
\]

The Kappa measures introduced by Kaplan and Knowles (2004) are defined by:

\[
Kappa_l(q) = \frac{\mathbb{E}_P[X] - q}{\left[\mathbb{E}_P[(q - X)^+]\right]^{1/l}}, \text{ for } l = 1, 2, \ldots,
\]

For \( l = 1 \), we get the Sharpe Omega measure and, for \( l = 2 \), we recover the Sortino ratio. Note also that Zakamouline (2010) shows that Kappa measures are performance measures based on piecewise linear plus power utility functions.

By means of Monte Carlo experiments, we simulate the Omega of a 2x and a 3x leveraged ETF as well as the Omega of the corresponding leveraged CPPI strategies; as in the previous section. We use the same set of parameters for the financial market.

In Table 7, we report the results of the simulations. We study the effect of the threshold on the ranking of the strategies according to the Omega performance measure is related to the maximization of an expected utility with loss aversion, as introduced by Tversky and Kahneman (1992).
Omega criterion. We find again confirmation that the leveraged CPPI strategy offers to investors more downside protection than LETF. Indeed, it is for low level of the threshold that LCPII strategies have higher Omega values. Again, this effect is all the more marked as the values of the multiple $m$ and of the floor parameter $\xi$ are high. Thus, it is when investors are more concerned about protecting their capital that LCPII strategies exhibit higher Omega values than LETF strategy. Notice that the LCPII 2 is never a dominant strategy. Once the threshold value is higher, LETF is dominant with respect to Omega. Note though that LETF and LCPII with $\xi = 50\%$ are very close.

Table 7. – OMEGA of LETF and LCPII Strategies for $L = 2$ and $L = 3$

<table>
<thead>
<tr>
<th>Threshold</th>
<th>LETF</th>
<th>LCPII</th>
<th>LCPII1</th>
<th>LCPII2</th>
<th>LETF</th>
<th>LCPII</th>
<th>LCPII1</th>
<th>LCPII2</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.50%</td>
<td>2.1814</td>
<td>2.1953</td>
<td>2.2011</td>
<td>2.1676</td>
<td>2.0803</td>
<td>2.0941</td>
<td>2.0862</td>
<td>1.9834</td>
</tr>
<tr>
<td>1.00%</td>
<td>2.1157</td>
<td>2.1252</td>
<td>2.1272</td>
<td>2.0859</td>
<td>2.0368</td>
<td>2.0483</td>
<td>2.0373</td>
<td>1.9295</td>
</tr>
<tr>
<td>1.50%</td>
<td>2.0484</td>
<td>2.0577</td>
<td>2.0562</td>
<td>2.0079</td>
<td>1.9943</td>
<td>2.0038</td>
<td>1.9899</td>
<td>1.8775</td>
</tr>
<tr>
<td>2.00%</td>
<td>1.9855</td>
<td>1.9927</td>
<td>1.9880</td>
<td>1.9332</td>
<td>1.9530</td>
<td>1.9605</td>
<td>1.9438</td>
<td>1.8272</td>
</tr>
<tr>
<td>2.50%</td>
<td>1.9249</td>
<td>1.9301</td>
<td>1.9224</td>
<td>1.8618</td>
<td>1.9127</td>
<td>1.9184</td>
<td>1.8990</td>
<td>1.7785</td>
</tr>
<tr>
<td>3.00%</td>
<td>1.8664</td>
<td>1.8698</td>
<td>1.8594</td>
<td>1.7935</td>
<td>1.8735</td>
<td>1.8774</td>
<td>1.8555</td>
<td>1.7315</td>
</tr>
<tr>
<td>3.50%</td>
<td>1.8099</td>
<td>1.8117</td>
<td>1.7988</td>
<td>1.7281</td>
<td>1.8353</td>
<td>1.8374</td>
<td>1.8133</td>
<td>1.6830</td>
</tr>
<tr>
<td>4.00%</td>
<td>1.7555</td>
<td>1.7558</td>
<td>1.7405</td>
<td>1.6655</td>
<td>1.7980</td>
<td>1.7986</td>
<td>1.7722</td>
<td>1.6419</td>
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<tr>
<td>4.50%</td>
<td>1.7030</td>
<td>1.7018</td>
<td>1.6844</td>
<td>1.6056</td>
<td>1.7617</td>
<td>1.7607</td>
<td>1.7323</td>
<td>1.5993</td>
</tr>
<tr>
<td>5.00%</td>
<td>1.6523</td>
<td>1.6498</td>
<td>1.6304</td>
<td>1.5481</td>
<td>1.7263</td>
<td>1.7239</td>
<td>1.6935</td>
<td>1.5581</td>
</tr>
<tr>
<td>5.50%</td>
<td>1.6034</td>
<td>1.5996</td>
<td>1.5784</td>
<td>1.4931</td>
<td>1.6918</td>
<td>1.6879</td>
<td>1.6558</td>
<td>1.5181</td>
</tr>
<tr>
<td>6.00%</td>
<td>1.5561</td>
<td>1.5512</td>
<td>1.5283</td>
<td>1.4404</td>
<td>1.6581</td>
<td>1.6530</td>
<td>1.6191</td>
<td>1.4794</td>
</tr>
</tbody>
</table>

We consider now the Kappa measure for $l = 2$, a Sortino type ratio. Table 8 reports the results of the simulations. Compared with Omega results, we notice that Kappa 2 measure leaves almost unchanged the ranking of the different strategies.

Table 8 reports the results of the simulations. Compared with Omega results, we notice that Kappa 2 measure leaves almost unchanged the ranking of the different strategies.

31. We perform the same simulations with a Kappa 3 measure. The ranking is the same as that obtained with Kappa 2 measure which is the Sortino ratio.
5. CONCLUDING REMARKS

In this paper, we analyze main properties of leveraged ETFs that correspond to convex constant allocation portfolio strategies. In particular, we illustrate how the stock index can experience an increase while at the same time the leveraged fund decreases, calculating the explicit probability of such an event for the GBM case. We also establish that, in continuous-time, a CPPI portfolio with a floor proportional to the value of the portfolio itself is also a convex constant allocation portfolio strategy. Taking a leveraged ETF issued by SGAM and managed under a slightly modified CPPI strategy that we call Leveraged CPPI, we propose a quasi explicit expression for its value. We compare both strategies by means of their first four moments as well as by applying Omega and Kappa performance measures. These measures are especially well designed for portfolios with asymmetric returns. Monte Carlo simulations allow us to point out the similarities in behavior of a standard leveraged ETF and of a Leveraged CPPI strategy. Nevertheless, we can conclude that the Leveraged CPPI strategy is essentially equivalent to a LETF with a little less expected return and a little more downside protection.

Table 8. – KAPPA 2 of LETF and LCPII Strategies for $L = 2$ and $L = 3$

<table>
<thead>
<tr>
<th>Threshold</th>
<th>LETF</th>
<th>LCPI</th>
<th>LCPII</th>
<th>LCPII</th>
<th>LETF</th>
<th>LCPI</th>
<th>LCPII</th>
<th>LCPII</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.50%</td>
<td>0.6691</td>
<td>0.6772</td>
<td>0.6618</td>
<td>0.6618</td>
<td>0.6608</td>
<td>0.6676</td>
<td>0.6620</td>
<td>0.6636</td>
</tr>
<tr>
<td>1.00%</td>
<td>0.6346</td>
<td>0.6414</td>
<td>0.6440</td>
<td>0.6426</td>
<td>0.6365</td>
<td>0.6422</td>
<td>0.6349</td>
<td>0.5733</td>
</tr>
<tr>
<td>1.50%</td>
<td>0.6010</td>
<td>0.6067</td>
<td>0.6074</td>
<td>0.5854</td>
<td>0.6138</td>
<td>0.6173</td>
<td>0.6084</td>
<td>0.5438</td>
</tr>
<tr>
<td>2.00%</td>
<td>0.5683</td>
<td>0.5729</td>
<td>0.5718</td>
<td>0.5458</td>
<td>0.5894</td>
<td>0.5930</td>
<td>0.5825</td>
<td>0.5150</td>
</tr>
<tr>
<td>2.50%</td>
<td>0.5365</td>
<td>0.5400</td>
<td>0.5372</td>
<td>0.5075</td>
<td>0.5666</td>
<td>0.5691</td>
<td>0.5571</td>
<td>0.4870</td>
</tr>
<tr>
<td>3.00%</td>
<td>0.5055</td>
<td>0.5080</td>
<td>0.5036</td>
<td>0.4705</td>
<td>0.5442</td>
<td>0.5457</td>
<td>0.5323</td>
<td>0.4596</td>
</tr>
<tr>
<td>3.50%</td>
<td>0.4753</td>
<td>0.4769</td>
<td>0.4709</td>
<td>0.4346</td>
<td>0.5222</td>
<td>0.5228</td>
<td>0.5080</td>
<td>0.4330</td>
</tr>
<tr>
<td>4.00%</td>
<td>0.4458</td>
<td>0.4466</td>
<td>0.4392</td>
<td>0.3999</td>
<td>0.5007</td>
<td>0.5004</td>
<td>0.4843</td>
<td>0.4070</td>
</tr>
<tr>
<td>4.50%</td>
<td>0.4171</td>
<td>0.4170</td>
<td>0.4083</td>
<td>0.3662</td>
<td>0.4796</td>
<td>0.4784</td>
<td>0.4610</td>
<td>0.3816</td>
</tr>
<tr>
<td>5.00%</td>
<td>0.3892</td>
<td>0.3883</td>
<td>0.3783</td>
<td>0.3336</td>
<td>0.4588</td>
<td>0.4569</td>
<td>0.4383</td>
<td>0.3569</td>
</tr>
<tr>
<td>5.50%</td>
<td>0.3620</td>
<td>0.3603</td>
<td>0.3492</td>
<td>0.3020</td>
<td>0.4385</td>
<td>0.4358</td>
<td>0.4160</td>
<td>0.3328</td>
</tr>
<tr>
<td>6.00%</td>
<td>0.3354</td>
<td>0.3331</td>
<td>0.3208</td>
<td>0.2714</td>
<td>0.4186</td>
<td>0.4151</td>
<td>0.3942</td>
<td>0.3092</td>
</tr>
</tbody>
</table>
REFERENCES


APPENDIX A: STATIC LEVERAGED STRATEGY

Under assumption (12) on the risky asset dynamics and with discrete-time rebalancing, we deduce:

\[
\frac{V_{SL}^t}{V_{SL}^0} = \exp \left[ \int_0^t \left( \mu_s - \frac{1}{2} \sigma_s^2 \right) ds + \int_0^t \sigma_s dW_s \right] - (L - 1)e^{r'}. \tag{40}
\]

Thus, for the GBM case, the random variable \( X = \frac{V_{SL}^t}{V_{SL}^0} + (L - 1)e^{r't} \) has a Lognormal distribution \( \text{LnN} \left( \text{Log} (L) + (\mu - \frac{1}{2} \sigma^2)t, \sigma \sqrt{t} \right) \).

The probability density function of \( \frac{V_{SL}^t}{V_{SL}^0} - 1 \) is plotted in Figure 8 for three different levels of leverage (\( L = 2, 3 \) and 5) and the same parameter values as in the main text are used:

![Figure 8](image-url)
APPENDIX B: SENSITIVITIES OF THE PROBABILITY $P_2$
THAT THE STOCK INDEX INCREASES WHILE THE LEVERAGED FUND DECREASES

Proof of Proposition 7

To determine the sensitivities of $P_2$, note that we have:

$$\frac{\partial a}{\partial \sigma} = L \left(1 - L\right) \sigma t; \quad \frac{\partial a}{\partial \mu} = 0; \quad \frac{\partial a}{\partial L} = -rt + \frac{1}{2} (1 - 2L) \sigma^2 t,$$

$$\frac{\partial a}{\partial r} = (1 - L) t; \quad \frac{\partial a}{\partial t} = r (1 - L) + \frac{1}{2} L \left(1 - L\right) \sigma^2.$$

i) Derivative of the probability $P_2$ with respect to the volatility $\sigma$.

We have:

$$\frac{\partial P_2}{\partial \sigma} = \frac{\partial}{\partial \sigma} \left( -\frac{a/L - (\mu - \frac{1}{2} \sigma^2) t}{\sigma \sqrt{t}} \right) \Phi' \left( \frac{-a/L - (\mu - \frac{1}{2} \sigma^2) t}{\sigma \sqrt{t}} \right)$$

$$= \left( - \frac{\partial}{\partial \sigma} \frac{a - \sigma^2 \sqrt{t} L}{\sigma^2 \sqrt{t}} + \frac{\mu \sqrt{t}}{\sigma^2} - a L \right) \Phi' \left( \frac{-a/L - (\mu - \frac{1}{2} \sigma^2) t}{\sigma \sqrt{t}} \right)$$

$$= \left( (L - 1 \sqrt{t} \left( \frac{1}{2} \sigma^2 \sqrt{t} L - r \right) L \sigma^2 \right) + \frac{\mu \sqrt{t}}{\sigma^2} + \frac{\sqrt{t}}{2} \Phi' \left( \frac{-a/L - (\mu - \frac{1}{2} \sigma^2) t}{\sigma \sqrt{t}} \right)$$

$$= \left( L^{-1} \sqrt{t} \left( \frac{1}{2} \sigma^2 \sqrt{t} \right) \right) + \frac{\mu \sqrt{t}}{\sigma^2} + \frac{\sqrt{t}}{2} \Phi' \left( \frac{-a/L - (\mu - \frac{1}{2} \sigma^2) t}{\sigma \sqrt{t}} \right)$$

ii) Derivative of the probability $P_2$ with respect to instantaneous expected return $\mu$.

Since $\frac{\partial a}{\partial \mu} = 0$, we have:

$$\frac{\partial}{\partial \mu} \left( \frac{-a/L - (\mu - \frac{1}{2} \sigma^2) t}{\sigma \sqrt{t}} \right) = \frac{\partial}{\partial \mu} \left( \frac{-a/L - (\mu - \frac{1}{2} \sigma^2) t}{\sigma \sqrt{t}} \right) = -\frac{1}{\sigma \sqrt{t}},$$

from which we deduce the result.
iii) Determination of the derivative of the probability $P_2$ with respect to the leverage $L$:

We have:

$$\frac{\partial P_2}{\partial L} = \frac{\partial}{\partial L} \left( \frac{-a/L - (\mu - \frac{1}{2}\sigma^2) t}{\sigma \sqrt{t}} \right) \Phi' \left( \frac{-a/L - (\mu - \frac{1}{2}\sigma^2) t}{\sigma \sqrt{t}} \right).$$

Then, from relation $\frac{\partial a}{\partial L} = -rt + \frac{1}{2}(1 - 2L)\sigma^2 t$, we deduce the result.

iv) Derivative of the probability $P_2$ with respect to the interest rate $r$:

We have:

$$\frac{\partial P_2}{\partial r} = -\frac{1}{\sigma L \sqrt{t}} \frac{\partial a}{\partial r} \Phi' \left( \frac{-a/L - (\mu - \frac{1}{2}\sigma^2) t}{\sigma \sqrt{t}} \right).$$

Then, using $\frac{\partial a}{\partial r} = (1 - L) t$, we get the result.

v) Derivative of the probability $P_2$ with respect to the time to maturity $t$:

We have:

$$\left( -\frac{1}{\sigma L \sqrt{t}} \frac{\partial a}{\partial t} \left( \frac{a}{\sqrt{t}} \right) - \frac{\mu - \frac{1}{2}\sigma^2}{2\sigma \sqrt{t}} \right) \Phi' \left( \frac{-a/L - (\mu - \frac{1}{2}\sigma^2) t}{\sigma \sqrt{t}} \right) + \frac{(\mu - \frac{1}{2}\sigma^2)}{2\sigma \sqrt{t}} \Phi' \left( \frac{-(\mu - \frac{1}{2}\sigma^2) t}{\sigma \sqrt{t}} \right).$$

Therefore, using $\frac{\partial a}{\partial t} = r (1 - L) + \frac{1}{2} L (1 - L) \sigma^2$, the result is proved.

**Proof of Corollary 8**

i) Condition $\frac{\partial P_2}{\partial \sigma} = 0$ is satisfied if and only if we have:

$$\frac{\Phi' \left( \frac{-a/L - (\mu - \frac{1}{2}\sigma^2) t}{\sigma \sqrt{t}} \right)}{\Phi' \left( \frac{-\mu - \frac{1}{2}\sigma^2} {\sigma \sqrt{t}} \right)} = \frac{(\mu \sqrt{t}/\sigma^2 + \sqrt{t})}{(L-1)\sqrt{t}(\frac{1}{2}L\sigma^2-r)} + \frac{\mu \sqrt{t} + \sqrt{t}}{\sigma^2}. $$
Since \( L > 1 \), the last term of this equation is non-negative. Thus, a solution \( \sigma_0 \) can exist as shown in numerical example.

\( \text{ii) Note that, since } L > 1, \text{ we have } a < 0. \text{ Additionally, since } \Phi'(x) = \frac{e^{-x^2/2}}{\sqrt{2\pi}}, \text{ we get the following equivalence:} \\
\frac{\partial P_2}{\partial \mu} > 0 \iff \left[ \frac{a}{L} + \left( \mu - \frac{1}{2}\sigma^2 \right)t \right]^2 > \left[ \left( \mu - \frac{1}{2}\sigma^2 \right)t \right]^2.
\]

This leads to the condition: \( (a/L) \left[ a/L + 2 \left( \mu - \frac{1}{2}\sigma^2 \right)t \right] > 0. \) Thus,
\[
\frac{\partial P_2}{\partial \mu} > 0 \iff \frac{a}{L} + 2 \left( \mu - \frac{1}{2}\sigma^2 \right)t < 0.
\]

Finally, we get a condition which corresponds to:
\[
(r \left[ 1 - \frac{1}{L} \right]/2 + \sigma^2 \left[ L + 1 \right])/4 > \mu.
\]

Note that this condition does not depend on time to maturity \( t \). As shown in next numerical example, the derivative \( \frac{\partial P_2}{\partial \mu} \) can be equal to 0 for standard value of \( \mu \) (in the example, \( \mu \simeq 4.9\% \)). It means that the probability \( P_2 \) can be an increasing then decreasing function of the instantaneous expected return \( \mu \).

\( \text{iii) } \frac{\partial P_2}{\partial L} > 0. \)

\( \text{iv) } \frac{\partial P_2}{\partial r} > 0 \text{ since } L > 1. \)

\( \text{v) If } (\mu - \frac{1}{2}\sigma^2) > 0 \text{ (standard case), a sufficient condition to get } \frac{\partial P_2}{\partial r} > 0 \text{ is that} \\
\left( \left( 1 - \frac{1}{L} \right)r + \frac{1}{2}L\sigma^2 - \mu \right) > 0.
\]

This leads to a second-order polynomial equation with respect to \( L \):
\[
\frac{1}{2}L^2\sigma^2 + L \left[ r - \mu \right] - r = 0.
\]
with a discriminant \([r - \mu]^2 + 2\sigma^2 r > 0\). Thus, an interior solution can exist but the only positive solution is \(L_1 = \frac{(\mu - r) + \sqrt{[\mu - r]^2 + 2\sigma^2 r}}{\sigma} > 0\).

(For the standard case \(\mu > r\)). The condition \(L_1 > 1\) is equivalent to \((\mu - r) + \frac{\alpha^2}{\sigma^2} > \frac{\alpha^2}{2}\). For the next numerical example, \(L_1 = 3\). Thus, since we consider the case \(L = 2\), the probability \(P_2\) is increasing with respect to the time to maturity \(t\).

**APPENDIX C: QUASI-EXPlicit SOLUTION FOR LEVERAGED CPPI FUND**

Assume now that the portfolio manager trades in continuous-time during each period \([t_n, t_{n+1}]\). Suppose also that the CPPI strategy is not capped. Then, the portfolio value can be determined by induction as follows:

Recall that the standard CPPI strategy, with a deterministic floor \(F_t\), induces a portfolio value defined by:

\[
V_{t}^{CPPI} = F_0 e^{r \tau} + \alpha_t \left( \frac{S_t}{S_0} \right)^m, \tag{42}
\]

where \(\alpha_t = C_0 \exp[\beta_t]\) and \(\beta_t = r(1 - m)t - (m^2 - m) \int_0^t \frac{\sigma^2 u^2}{2} du\).

Consequently, applying relation (42) for each management period \([t_n, t_{n+1}]\), we can determine the portfolio value of the leveraged CPPI:

Denote for all \(s < t\),

\[
\alpha_{s,t} = C_0 \exp[\beta_{s,t}] \quad \text{and} \quad \beta_{s,t} = r(1 - m)(t - s) - (m^2 - m) \int_0^t \frac{\sigma^2 u^2}{2} du.
\]

We have:

For \(t \in [0, t_1], V_t = F_0 e^{r \tau} + \alpha_t \left( \frac{S_t}{S_0} \right)^m, \)

For \(t \in [t_1, t_2], V_t = \xi V_{t_1} e^{r(t-t_1)} + \alpha_{t_1,t} \left( \frac{S_t}{S_{t_1}} \right)^m, \)

with \(\alpha_{t_1,t} = (1 - \xi) V_{t_1} \exp[\beta_{t_1,t}]\).
For $t \in [t_n, t_{n+1}]$, $V_t = \xi V_{t_n} e^{\gamma (t - t_n)} + \alpha_{t_n, t} \left( \frac{S_t}{S_{t_n}} \right)^m$, with

$$
\alpha_{t_n, t} = (1 - \xi) V_{t_n} \exp \left[ \beta_{t_n, t} \right] .
$$

Then, we deduce: If the leveraged CPPI is continuously rebalanced and not capped, then its value is given by: for any $t \in [t_n, t_{n+1}]$,

$$
V_t = V_0 \left[ \prod_{1 \leq k \leq n} \left( \xi e^{\gamma (t_k - t_{k-1})} + (1 - \xi) e^{\beta_{t_{k-1}, t_n} \left( \frac{S_{t_k}}{S_{t_{k-1}}} \right)^m} \right) \right] \times \left( \xi e^{\gamma (t - t_n)} + (1 - \xi) \exp \left[ \beta_{t_n, t} \right] \left( \frac{S_t}{S_{t_n}} \right)^m \right).
$$

Consider for instance the geometric Brownian case. Assume also to simplify that durations $(t_{n+1} - t_n)$ are equal to a constant $\delta$ and that portfolio maturity $T$ is equal to $t_N$. Then, the ratio $\frac{S_{t_k}}{S_{t_{k-1}}}$ is given by:

$$
\left( \frac{S_{t_k}}{S_{t_{k-1}}} \right)^m = \exp \left( m \left[ (\mu - 1/2\sigma^2) \delta + \sigma (W_{t_k} - W_{t_{k-1}}) \right] \right).
$$

i) The portfolio value $V_T$ has a quasi-explicit form:

$$
V_T = V_0 \left[ \prod_{1 \leq k \leq N} \left( \xi e^{\gamma \delta} + (1 - \xi) e^{\beta_{t_n} \left( \frac{S_{t_k}}{S_{t_{k-1}}} \right)^m} \right) \right].
$$

Note that the rate of return on the period $[t_n, t_{n+1}]$ is given by:

$$
(1 - \xi) \exp \left[ \beta_{t_n} \right] \frac{S_{t_{n+1}}}{S_{t_{n-1}}} - \frac{S_{t_n}}{S_{t_{n-1}}} + \xi e^{\beta_{t_n}} - 1.
$$

ii) The probability distribution of portfolio return can be determined as follows: The random variables $\left( \frac{S_{t_k}}{S_{t_{k-1}}} \right)$ are i.i.d. with Lognormal distribution. Thus, the logreturn of the portfolio $\ln \left( \frac{V_T}{V_0} \right)$ is the sum of $N$ i.i.d. random variables. Their common probability distribution is an affine transformation of the Lognormal distribution with parameters
Thus, the Pdf of the logreturn of the portfolio $\ln (V_T / V_0)$ is given by\textsuperscript{32}:

$$f_{\ln(V_T / V_0)}(x) = \ast_{1 \leq \gamma \leq N} g_n(x),$$

where $g_n(x)$ is given by:

$$g_n(x) = \frac{1}{\sqrt{2\pi}} \times \frac{1}{m \sigma \sqrt{\delta}} \exp \left[ - \frac{\left( \ln \left( \frac{\gamma - \xi}{\xi} \right) - m(\mu - 1/2\sigma^2)\delta \right)^2}{2m^2\sigma^2\delta} \right] \mathbb{1}_{x > \xi}.$$

\textsuperscript{32} The symbol $\ast$ denotes the convolution product.